TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 321, Number 1, September 1990

IDENTITIES ON QUADRATIC GAUSS SUMS

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ABSTRACT. Given a local field F, each multiplicative character θ of the split algebra $F \times F$ or of a separable quadratic extension of F has an associated generalized Gauss sum γ_θ^F. It is a complex valued function on the character group of $F^\times \times F$, meromorphic in the first variable. We define a pairing between such Gauss sums and study its properties when F is a nonarchimedean local field. This has important applications to the representation theory of $GL(2, F)$ and correspondences [GL3].

INTRODUCTION

The multiplicative group $F^\times$ of a local field F is a split extension of the value group $|F^\times|$ by the compact group of the units. Hence, the group $\mathscr{A}(F^\times)$ of continuous homomorphisms of $F^\times$ in $\mathbf{C}^\times$ is a one-dimensional complex Lie group, with connected component of identity the image of $\mathbf{C}$ under the map

$$s \mapsto (t \mapsto |t|^s).$$

We have written $|t|$ for the normalized absolute value of t.

The group $F^\times$ acts on functions on F by translations:

$$t: f \mapsto f^t, \quad f^t(x) = f(tx), \qquad t \in F^\times.$$

This gives an action of $F^\times$ on the space $\mathscr{S}(F)$ of Schwartz-Bruhat functions on F, hence also an action on the space $\mathscr{S}'(F)$ of tempered distributions on F:

$$\langle t.D | f^t \rangle = \langle D | f \rangle.$$

For each χ in $\mathscr{A}(F^\times)$, the space of tempered distributions of type χ under the action of $F^\times$ is one-dimensional (e.g. [W3]). The choice of a nontrivial additive unitary character ψ of F defines an identification of F with its Pontrjagin dual by $(u, v) \mapsto \psi(uv)$, hence a self-dual Haar measure $d_\psi u$ on F, and a Fourier transform

$$\hat{f}(v) = \int_F f(u)\psi(uv)\, d_\psi u, \qquad f \in \mathscr{S}(F).$$

Received by the editors July 10, 1988 and, in revised form, November 21, 1988.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11L05; Secondary 11S37, 33A15.

Key words and phrases. Gauss sum, quadratic gamma factor.

Both authors supported in part by NSF Grant DMS-8404083.

The Fourier transform exchanges distributions of type χ with distributions of type $t \mapsto |t|\chi(t)^{-1}$.

Fix an additive Haar measure du on F and let d^*u be the measure $|u|^{-1/2}du$ on F. Then, for each $\chi \in \mathscr{A}(F^\times)$, we have a measure χd^* on $F^\times$, which is holomorphic in χ. It is well known [W3] that χd^* extends to a unique meromorphic distribution Δ_χ of type $\chi|\ |^{1/2}$ on F, which has simple poles. As $\hat{\Delta}_\chi$ is a multiple of $\Delta_{\chi^{-1}}$, denote their ratio by $\gamma^F(\chi, \psi)$, called the gamma factor attached to χ and ψ:

$$\hat{\Delta}_\chi = \gamma^F(\chi, \psi)\Delta_{\chi^{-1}}.$$

It is a meromorphic function of χ, satisfying the complement formula

$$\gamma^F(\chi, \psi)\gamma^F(\chi^{-1}, \psi^{-1}) = 1,$$

and also

$$\gamma^F(\chi, \psi^t) = \chi(t)^{-1}\gamma^F(\chi, \psi), \qquad t \in F^\times,$$
$$\gamma^F(1, \psi) = 1.$$

In case F is a nonarchimedean local field, the gamma factor $\gamma^F(\chi, \psi)$ is a Gauss sum; more precisely, it is an analytic continuation of the following integral taken in principal value:

$$\int_F \chi(u)\psi(u)d^*_\psi u.$$

Let K be a quadratic étale algebra over F, that is, K is either an F-algebra isomorphic (in two ways) to $F \times F$, or a separable quadratic field extension of F. Then the norm group $N_{K/F}(K^\times)$ has index 1 or 2 according as K splits or not over F. Denote by $\eta_{K/F}$ the character of $F^\times$ with kernel $N_{K/F}(K^\times)$. For a character θ of $K^\times$, a character χ of $F^\times$, and ψ as above, we define the quadratic Gauss sum with $T_{K/F}$ the trace form from K to F:

$$\gamma^F_\theta(\chi, \psi) = \lambda_{K/F}(\psi)\gamma^K(\theta\chi \circ N_{K/F}, \psi \circ T_{K/F}),$$

where

$$\lambda_{K/F}(\psi) = \gamma^F(\eta_{K/F}, \psi).$$

Note that in case K is split over F, the group $\mathscr{A}(K^\times)$ is isomorphic to the product of two copies of $\mathscr{A}(F^\times)$ so that we may identify θ with a couple $\{\mu, \nu\}$ of characters of $F^\times$; in this case, we have $\lambda_{K/F}(\psi) = 1$, and

$$\gamma^F_\theta(\chi, \psi) = \gamma^F(\chi\mu, \psi)\gamma^F(\chi\nu, \psi).$$

It is known that these quadratic Gauss sums have the following properties:
(a) if the additive character ψ is changed to ψ^t, $t \in F^\times$, then

$$\gamma^F_\theta(\chi, \psi^t) = \chi(t)^{-2}\omega(t)^{-1}\gamma^F_\theta(\chi, \psi), \qquad \omega(t) = \eta_{K/F}(t)\theta(t);$$

(b) it satisfies the Davenport-Hasse identity, namely, when $\theta = \mu \circ N_{K/F}$ for some character μ of $F^\times$, one has

$$\gamma_\theta^F(\chi, \psi) = \gamma^F(\chi\mu, \psi)\gamma^F(\chi\mu\eta_{K/F}, \psi);$$

(c) for F nonarchimedean and χ of conductor large enough

$$\gamma_\theta^F(\chi, \psi) = \gamma^F(\chi, \psi)\gamma^F(\chi\omega, \psi),$$

where ω is as in (a); this is the deep twist property.

In this article, we shall study, for F nonarchimedean, a pairing between two such quadratic Gauss sums γ_θ^F and $\gamma_{\theta'}^F$ relative to two quadratic étale algebras K and K'. It is defined for the two meromorphic functions

$$\chi \mapsto \gamma_\theta^F(\chi, \psi) \quad \text{and} \quad \chi \mapsto \gamma_{\theta'}^F(\chi^{-1}, \psi)$$

having no common pole as the finite part of a contour integral over $\mathscr{A}(F^\times)$ enclosing the poles of $\gamma_{\theta'}^F(\chi^{-1}, \psi)$ but no poles of $\chi \mapsto \gamma_\theta^F(\chi, \psi)$:

$$\langle \gamma_\theta^F | \gamma_{\theta'}^F \rangle_\psi = \oint_{\mathscr{A}(F^\times)} \gamma_\theta^F(\chi, \psi)\gamma_{\theta'}^F(\chi^{-1}, \psi)\, d\chi.$$

Our purpose is to derive further properties of the quadratic Gauss sums from this pairing, with the goal of reestablishing Langland's correspondences on representations of degree two semisimple algebras over F. Thus, some of the results would become immediate consequences if one were to grant these correspondences.

Our main result (Theorem 1) expresses the value of this pairing in terms of a gamma factor coming from the étale F-algebra $B = K \otimes_F K'$ and the character $\theta \times \theta'$ of $B^\times$ defined by

$$(\theta \times \theta')(z) = \theta \circ N_{B/K}(z)\theta' \circ N_{B/K'}(z).$$

In Theorem 2, we show that for K' split the formula reduces to a formula which has appeared already in [L, GL1, GL2], called the multiplicative formula for γ_θ^F. When K and K' are not isomorphic and when the product of the restriction to $F^\times$ of θ and θ' is the character $t \mapsto |t|^{-1}$, the value of the pairing $\langle \gamma_\theta^F | \gamma_{\theta'}^F \rangle_\psi$ can be simplified (§3.3). In particular, it depends only on the fields K and K'. This fact is used in [GL3] to characterize the degree two monomial representations of the local Weil group W_F over F. Each character θ of $K^\times$ determines a two-dimensional representation $\mathrm{Ind}_K^F \theta$ of W_F. We prove in §3.4 that the quadratic Gauss sums γ_θ^F parametrize the isomorphism classes of these representations $\mathrm{Ind}_K^F \theta$.

1. Preparation

1.1. We introduce some notations for any nonarchimedean field. In general, the field will be indicated by a subscript, but it will be deleted for the base field

F. The ring of integers of F is $\mathscr{O} = \mathscr{O}_F$, its group of units is $\mathscr{O}^\times = \mathscr{O}_F^\times$, its maximal ideal is $\mathscr{P} = \mathscr{P}_F$. As $\mathscr{O}$ and $\mathscr{O}^\times$ are open compact subgroups of F, $F^\times$, respectively, we choose Haar measures $du = d_F u$ on F, $d^\times t = d_F^\times t$ on $F^\times$, respectively, giving to them the volume 1. Then $d^\times t = L_F |t|^{-1}\, dt$, where $L_F = (1 - q^{-1})^{-1}$ is the value at the character $t \to |t|$ of the L-function of F, and $q = q_F$ is the module of F.

On the group $\widehat{F}$ of characters of F, there is an absolute value $|\ |$ defined on $\psi \in \widehat{F}$ as the smallest number c in the value group of F such that $\psi(u) = 1$ for $c|u| \leq 1$. Then $|\psi^t| = |\psi||t|$ for t in F. The self-dual Haar measure on F associated to the bicharacter $\psi(uv)$, for ψ nontrivial in $\widehat{F}$, is $d_\psi u = |\psi|^{1/2}\, du$.

We define a modification $\Gamma^F = \Gamma$ of the gamma factor γ^F by taking the finite part of the following integral:

$$\Gamma(\chi, \psi) = \int_F \chi(t)\psi(t) d^\times t.$$

It is given in terms of γ^F by

$$\Gamma(\chi, \psi) = L_F |\psi|^{-1/2} \gamma^F(\chi q^{1/2}, \psi)$$

and satisfies the following complement formula:

$$\Gamma(\chi, \psi)\Gamma(\chi^{-1} q^{-1}, \psi) = L_F^2 |\psi|^{-1} \chi(-1).$$

Here, we have used the convention which identifies a nonzero complex number Z with the character $t \mapsto Z^{\operatorname{ord} t}$ of $F^\times$, where $\operatorname{ord} t = -\log_q |t|$.

For a character χ of $F^\times$, we write $a(\chi)$ for its conductor, and $A(\chi) = q^{a(\chi)}$; so, $A(\chi)$ is the smallest number $c \geq 1$ such that, for t a unit, we have $\chi(t) = 1$ when $c|t-1| \leq 1$, i.e. $\operatorname{ord}(t-1) \geq a(\chi)$ means $|t-1|A(\chi) \leq 1$. We define also $A'(\chi)$ to be $A(\chi)$ if χ ramifies and to be q otherwise. We denote by $|\chi|$ the character $t \mapsto |\chi(t)|$, so that it coincides with $|Z|$ when χ is given by the nonzero complex number Z.

For $|\chi| < q^{1/2}$ the gamma factor $\gamma^F(\chi, \psi)$ is given by the convergent integral

$$\gamma^F(\chi, \psi) = \int_{|t| \leq A'(\chi)|\psi|^{-1}} \chi(t)\psi(t) d_\psi^* t,$$

with $d_\psi^* t = |t|^{-1/2} d_\psi t$. When χ ramifies, only the shell $|t| = A(\chi)|\psi|^{-1}$ contributes; moreover, there is χ_ψ in this shell such that, for any character μ of $F^\times$ satisfying $A(\mu)^2 \leq A(\chi)$,

$$\gamma^F(\chi\mu, \psi) = \mu(\chi_\psi)\gamma^F(\chi, \psi).$$

From this, it follows that for $K = F \times F$ and θ a character of $K^\times$ given by the characters μ and ν of $F^\times$, we have the relation

$$\gamma_\theta^F(\chi, \psi) = \gamma^F(\chi, \psi)\gamma^F(\chi\omega, \psi) \quad \text{if } A(\mu)^2 \text{ and } A(\nu)^2 \leq A(\chi), \quad \omega = \mu\nu.$$

1.2. For K an étale F-algebra, that is, a product of the extensions E of F, its ring of integers $\mathcal{O}_K$ is the product of the $\mathcal{O}_E$'s, and the group $\mathcal{O}_K^\times$ of units is the product of the $\mathcal{O}_E^\times$'s. Let $T = T_{K/F}$ be the trace form from K to F. Then the pairing $(x, y) \mapsto T(xy)$ from $K \times K$ to F is nondegenerate. By composition with $F \to F/\mathcal{O}_F$, we get an orthogonality relation between the $\mathcal{O}_K$- submodules of K. The index of $\mathcal{O}_K$ in its orthogonal is called the discriminant $D_{K/F}$: it is the product of the $D_{E/F}$'s. The self-dual Haar measure $d_{K,\psi}$ on K associated to the bicharacter $\psi \circ T(xy)$ is $|\psi \circ T|^{1/2} d_K$, with $|\psi \circ T|$ the product of the $|\psi \circ T_{E/F}|$'s; hence $d_{K,\psi}$ is the product of the $d_{E,\psi}$'s. From Corollary 3 to Proposition 4 of Chapter VIII-1 in [W2], we have $|\psi \circ T| = |\psi|^{[K:F]} D_{K/F}^{-1}$. Let $N = N_{K/F}$ be the norm map on K: it is the product of the norm maps $N_{E/F}$. We introduce the two numbers

$$L_K = \prod_E L_E = \prod_E (1 - q_E^{-1})^{-1} \quad \text{and} \quad L_{K/F} = L_K / L_F^{[K:F]}.$$

Note that L_K is also the integral over $\mathcal{O}_K$ of the function $|Nx|$ for the measure $d_K^\times$. We denote by $|x|_K$ the absolute value $\mathrm{Max}_E |N_{E/F} x_E|$ on K, $x = (x_E)$.

1.3. A quadratic étale F-algebra K has a conjugation $^-$; its norm $N = N_{K/F}$ and trace $T = T_{K/F}$ are also given by $Nx = x\overline{x}$, $Tx = x + \overline{x}$. For each nontrivial additive character ψ of F, the quadratic character $\psi \circ N$ is nondegenerate and defines a fourth roots of unity $\lambda(\psi \circ N)$ by the functional equation [W1, G]:

$$\int_K \hat{f}(y) \psi \circ N(y)\, dy = \lambda(\psi \circ N) \int_K f(x) \psi^{-1} \circ N(x)\, dx$$

where f lies in the Schwartz-Bruhat space $\mathcal{S}(K)$ of compactly supported locally constant functions on K, and its Fourier transform is

$$\hat{f}(y) = \int_K f(x) \psi \circ T(x\overline{y}) d_{K,\psi} y.$$

With f the characteristic function of a sufficiently small ball around 0, we get

$$\lambda(\psi \circ N) = \int_{|x|_K \le R} \psi \circ N(x) d_{K,\psi} x, \quad \text{for } R \text{ large enough}.$$

Moreover, decomposing F with respect to the norm group of K, we get

$$\lambda(\psi \circ N) = \gamma^F(\eta_{K/F}, \psi),$$

with the notations as in the introduction. Note that $\lambda(\psi \circ N) = 1$ for K split.

1.4. If K is a quadratic étale F-algebra and θ is a character of $K^\times$, we define a two-dimensional representation of the Weil group W_F of F as follows. If K a field, its Weil group W_K appears as the kernel of the composition of the class field theory map $W_F \to F^\times$ with $\eta_{K/F}$; it has index two in W_F and θ gives a one-dimensional representation of W_K from the map $W_K \to K^\times$, hence by induction a two-dimensional representation $\mathrm{Ind}_K^F \theta$ of W_F. If K is split, then θ is given by two characters μ and ν of $F^\times$; in this case $\mathrm{Ind}_K^F \theta$ is the sum of the two one-dimensional representations of W_F defined by μ and ν.

Lemma. *Assume that K is a separable quadratic extension of F. Given a character θ of $K^\times$ and two characters α and β of $F^\times$, the following conditions are equivalent*:

(i) $\theta = \alpha \circ N_{K/F} = \beta \circ N_{K/F}$, $\beta = \alpha\eta_{K/F}$,

(ii) $\mathrm{Ind}_K^F \theta = \alpha \oplus \beta$,

(iii) $\gamma_\theta^F(\chi, \psi) = \gamma^F(\chi\alpha, \psi)\gamma^F(\chi\beta, \psi)$ *for any character χ of $F^\times$.*

Proof. (i) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (i) comes from the fact that $\mathrm{Ind}_K^F \theta$ is reducible if and only if the character θ is fixed under the conjugation of K over F. (i) $\Rightarrow$ (iii) is the Davenport-Hasse identity. Finally (iii) $\Rightarrow$ (i) uses the fact that θ is not regular since γ_θ^F has poles, so $\theta = \alpha \circ N_{K/F}$ and $\gamma_\theta^F(\chi, \psi)$ is $\gamma^F(\chi\alpha, \psi)\gamma^F(\chi\alpha\eta_{K/F}, \psi)$ with poles at $\alpha^{-1}q^{-1/2}$ and $\alpha^{-1}\eta_{K/F}q^{-1/2}$, which are also $\alpha^{-1}q^{-1/2}$ and $\beta^{-1}q^{-1/2}$: this gives the implication.

1.5. Lemma. *Let K be a quadratic F-algebra, with norm N, trace T, discriminant D, absolute value $|\ |_K$. Fix a nontrivial additive character ψ of F. If the element R of $|K^\times|_K$ satisfies $R|\psi| \geq D$, then, the Fourier transform with respect to $\psi \circ T$ of the function g on K defined by*

$$g(x) = \psi(-Nx) \text{ for } |x-1|_K \leq R \text{ and } 0 \text{ otherwise}$$

is equal to $\lambda_{K/F}(\psi)^{-1}\overline{g}$.

Proof. We write the inequality $R|\psi| \geq D$ as $R^{-1}|\psi \circ T|^{-1}$; this gives $\psi(Nx) = 1$ for $|x|_K \leq R^{-1}|\psi \circ T|^{-1}$; the orthogonal of this subgroup with respect to the bicharacter $\psi \circ T(x\overline{y})$ is $|y|_K \leq R$. From the reduction theorem in [G], the λ-factor of the quadratic group $(K, \psi \circ N)$ is also the λ-factor of the factor group of the ideal $|x|_K \leq R$ modulo the ideal $|x|_K \leq R^{-1}|\psi \circ T|^{-1}$ for the quadratic character $\psi \circ N$, hence

$$\lambda_{K/F}(\psi) \int_{|x|_K \leq R} f(x)\psi \circ N(x)^{-1}\, dx = \int_{|x|_K \leq R} \hat{f}(y)\psi \circ N(y)\, dy,$$

for all f in $\mathscr{S}(K)$. We write now the value at $1 + z$ of the Fourier transform of the given function g: $\hat{g}(1+z) = \psi(1+Tz)\int_{|x|_K \leq R} \psi(-Nx + Txz) d_{K,\psi}x$. We apply then the above formula to the function $f(x) = \psi(Txz)$ for $|x|_K \leq R$ and 0 elsewhere:

$$\hat{g}(1+z) = \psi(1+Tz)\lambda_{K/F}(\psi)^{-1} \int_{\substack{|x-z|_K \leq R^{-1}|\psi\circ T|^{-1} \\ |x|_K \leq R}} \psi(Nx) d_{K,\psi}x \int_{|y|_K \leq R} d_{K,\psi}y.$$

By integrating first on the ball $|x|_K \leq R^{-1}|\psi \circ T|^{-1}$, the first integral is seen to be 0 unless $|z|_K \leq R$; in this case, we get

$$\hat{g}(1+z) = \psi(1+Tz)\lambda_{K/F}(\psi)^{-1}\psi(Nz) = \lambda_{K/F}(\psi)^{-1}\psi \circ N(1+z),$$

which proves the lemma.

1.6. Biquadratic étale F-algebras. A biquadratic étale F-algebra B is a four-dimensional étale F-algebra containing at least two quadratic sub-F-algebras K and K'. Then, the map $(x, y) \mapsto xy$ gives an isomorphism from $K \otimes_F K'$ onto B. The conjugation of K (resp. K') with respect to F extends to an F-involution on B with K' (resp. K) as fixed points. These two involutions commute, and their composition has fixed points a third quadratic subalgebra K'', and K, K', K'' are all the three quadratic sub-F-algebras in B. When B is a field, it is a biquadratic extension of F, and K, K', K'' are the three quadratic extensions of F contained in B. When B contains exactly one split quadratic F-algebra, it is a direct product of two isomorphic quadratic separable extensions of F: we see B as $K \times K$ with the two subfields $\{(x, x) | x \in K\}$, $\{(x, \overline{x}) | x \in K\}$ and the split algebra $F \times F$ embedded naturally; in this case, the three involutions are respectively $(x, y) \mapsto (y, x)$, $(\overline{y}, \overline{x})$, $(\overline{x}, \overline{y})$, we have isomorphisms from $K \otimes_F K$ onto B and from $K \otimes_F (F \times F)$ onto B given by $x \otimes y \mapsto (xy, x\overline{y})$ and $x \otimes (u, v) \mapsto (xu, xv)$ respectively. Finally, when B contains more than one split quadratic F-algebra, then it is completely split, and isomorphic to F^4; we see then K, K', K'' as fixed points of the involutions $(t, u, v, w) \mapsto (v, w, t, u)$, (w, v, u, t), (u, t, w, v) respectively.

In each case, we have $\eta_{K/F}\eta_{K'/F}\eta_{K''/F} = 1$. This implies that the product $\lambda_{K/F}(\psi)\lambda_{K'/F}(\psi)\lambda_{K''/F}(\psi)$ is independent of the character ψ; we denote it by $\lambda_{B/F}$. If θ is a character of $B^\times$, we define for χ and ψ as above

$$\gamma_\theta^F(\chi, \psi) = \lambda_{B/F}\gamma^B(\theta\chi \circ N_{B/F}, \psi \circ T_{B/F}).$$

For the number $L_{B/F}$ defined in 1.2, we have

$$L_{B/F} = L_{K/F}L_{K'/F}L_{K''/F},$$

expressing the inductivity property of the L-function. The discriminant $D_{B/F}$ satisfies a similar relation ([S, Chapter VI.2] with the Artin representation, and [W2, Corollary 2 of Theorem 5, Chapter XII.4] with the Herbrand distribution):

$$D_{B/F} = D_{K/F}D_{K'/F}D_{K''/F}.$$

Also, we have the relations

$$\eta_{B/K} = \eta_{K'/F} \circ N_{K/F}.$$

In particular, the elements of F and those of $K_1 = \operatorname{Ker} N_{K/F}$ are norms from elements of B. We remark also that the conjugation of B over K' when restricted to K induces the conjugation of K over F. Finally, the product $\lambda_{B/F}\eta_{K/F}(-1)$ is the factor $\lambda_{B/K}(\psi \circ T_{K/F})$ for any ψ as above. We write it simply $\lambda_{B/K}$.

2. Preliminary results

2.1. Some measures. Let K be a degree n étale F-algebra, with its trace form T and its norm form N. The bilinear form $(x, y) \mapsto T(xy)$ on K is nondegenerate; hence, it defines a self-pairing on the top exterior power $\bigwedge^n K$, and a Haar measure $d_{K/F}$ on K. In other words, we have, by definition of the discriminant $D_{K/F}$,

$$\int_{\mathscr{O}_K} d_{K/F}x = D_{K/F}^{-1/2}. \tag{2.1}$$

The maps $x \mapsto x/Tx$ and $x \mapsto Tx$ give a decomposition of the complement in K of the hyperplane $\operatorname{Ker} T$ as the product of the affine hyperplane K_T consisting of trace 1 elements, by $F^\times$. This in turn yields a decomposition of the differential forms on K, hence a decomposition of the Haar measure $d_{K/F}x$ as $|Tx|^{n-1} d_T(x/Tx)\, dTx$, where $d_T y$ is a measure on K_T invariant under translations by $\operatorname{Ker} T$. This is also

$$\frac{d_{K/F}x}{|Nx|} = \frac{d_T(x/Tx)}{|N(x/Tx)|}\frac{dTx}{|Tx|}.$$

When passed to the normalized Haar measures on $K^\times$ and $F^\times$, this defines a measure $d^\bullet y$ on K_T by

$$d_K^\times x = d^\bullet(x/Tx) d_F^\times Tx.$$

In terms of the measure d_T introduced above and the number L_K in (1.2) this gives

$$d^\bullet y = D_{K/F}^{1/2} \frac{L_K}{L_F} \frac{d_T y}{|Ny|}.$$

When K is a field, this can also be stated as follows. The projective space deduced from K as an n-dimensional space over F is also the factor group $K^\times/F^\times$; the affine hyperplane K_T of K imbeds in $K^\times/F^\times$ as the open subset image by $x \mapsto x/Tx$ of the complement of the hyperplane $\operatorname{Ker} T$ of K; the Haar measure on $K^\times/F^\times$ given by the quotient of the normalized Haar measures on $K^\times$ and $F^\times$ induces on this open subset the measure $d^\bullet y$. It is a bounded measure since it gives to K_T the volume of $K^\times/F^\times$ with respect to the quotient measure $d_K^\times x/d_F^\times t$; this volume is computed by integrating the characteristic function of units in $K^\times$, which has volume 1: this gives

$$\int_{K_T} d^\bullet y = \int_{K^\times/F^\times} (d^\times x/d^\times t) = [|K^\times|_K : |F^\times|_K] = e_{K/F}$$

with $e_{K/F}$ the ramification index of F in K.

If K is a separable quadratic extension of F, with conjugation $x \mapsto \overline{x}$, then the homography $x \mapsto x^{-1} - 1$ of the projective line $K \cup \{\infty\}$ sends $K_T \cup \{\infty\}$ onto K_1 since $N(x^{-1} - 1) - 1 = (1 - Tx)/Nx$. As $y^{-1} - 1 = \overline{y}/y$ for $y \in K_T$, this map coincides on K_T with $x \mapsto \overline{x}/x$, which is a homomorphism from $K^\times$ onto K_1 (by Hilbert Theorem 90) with kernel $F^\times$. Hence it identifies the groups K_1 and $K^\times/F^\times$ in a compatible way with the two maps $K_T \to K^\times/F^\times$

and $K_T \to K_1$. This shows that, for f an integrable function on K_T with respect to the measure $d^\bullet y$,

$$\int_{K_T} f(y)d^\bullet y = e_{K/F}\int_{K_1} f((1+w)^{-1})d^\times w,$$

where $d^\times w$ is the normalized Haar measure on K_1 since the ramification index $e_{K/F}$ is the volume of K_T under $d^\bullet y$. If K is $F\times F$, then K_1 is isomorphic to $F^\times$ by $t\mapsto (t, t^{-1})$, and $x\mapsto x^{-1}-1$ sends $(K\cup\{\infty\})\setminus(\{0\}\times F)\cup(F\times\{0\})$ onto $F^\times$, and $(K_T\cup\{\infty\})\setminus\{(1,0),(0,1)\}$ bijectively onto $F^\times$ since $y^{-1}-1=\bar{y}/y$ for $y\in K_T$. In this case, for f an integrable function on K_T with respect to $d^\bullet y = dy_1/|y_1(1-y_1)|$ at $y=(y_1, y_2)$, we have

$$\int_{K_T} f(y)d^\bullet y = \int_{F^\times} f((1+t^{-1})^{-1}, (1+t)^{-1})d^\times t.$$

2.2. A formula for the gamma functions Γ^K.

Lemma. *Let K be a finite separable extension of F, and let K_T be the affine F-hyperplane of K consisting of elements y with trace $Ty=1$. If $\mu\in\mathscr{A}(K^\times)$ has the module of its restriction to $F^\times$ larger than q^{-1}, then μ is integrable on K_T with respect to the measure $d^\bullet y$ and*

$$\int_{K_T}\mu(y)d^\bullet y = \Gamma^K(\mu, \psi\circ T)/\Gamma(\mu|_{F^\times}, \psi).$$

The proof proceeds by analytic continuation from the case $q_K^{-1/n} < |\mu| < 1$, where n is the degree of K over F; this later case is straightforward.

2.3. The pairing $\langle\ ,\ \rangle_\psi$. Let $h(\chi, \psi)$ and $h'(\chi, \psi)$ be two rational functions of the characters χ of $F^\times$, depending on the nontrivial additive character ψ of F. We assume they satisfy the following conditions:

(a) there are numbers a, a' and $\omega, \omega'\in\mathscr{A}(F^\times)$ such that, for χ of large enough conductor,

$$h(\chi, \psi) = a\gamma^F(\chi, \psi)\gamma^F(\chi\omega, \psi),$$
$$h'(\chi, \psi) = a'\gamma^F(\chi, \psi)\gamma^F(\chi\omega', \psi);$$

(b) no pole of h is the inverse of a pole of h'.

We define then $\langle h|h'\rangle_\psi\in\mathbf{C}\cup\{\infty\}$ as follows. Observe that for $\mu\in\mathscr{A}(F^\times)$, the function $\chi\mapsto h(\chi\mu, \psi)$ satisfies a) with ω replaced by $\omega\mu^2$; except for a finite number of μ's, condition b) is satisfied for $h(\chi\mu, \psi)$ and $h'(\chi, \psi)$. We denote by $d\chi$ the 1-form on $\mathscr{A}(F^\times)$ read as $\frac{1}{2\pi i}Z^{-1}\,dZ$ on each connected component $\chi\mathbf{C}^\times$. Due to property (a), for n large enough, say, $n>N$, the integral

$$\oint_{a(\chi)=n} h(\chi Z, \psi)h'(\chi^{-1}, \psi)\,d\chi, \qquad Z\in\mathbf{C}^\times,$$

taken on simple positive contours around the origin in each component of conductor $a(\chi) = n$, is 0 if $\omega\omega'$ ramifies, and otherwise is

$$aa'\omega(-1) \oint_{a(\chi)=n} Z^{-2\,\mathrm{ord}\,\psi - 2n} (\omega\omega')^{-\mathrm{ord}\,\psi - n} \, d\chi$$
$$= aa'(1-q^{-1})^2 Z^{-2\,\mathrm{ord}\,\psi} (\omega\omega')^{-\mathrm{ord}\,\psi} (qZ^{-2}(\omega\omega')^{-1})^n ;$$

hence for $|Z^2\omega\omega'| > q$, the series

$$\sum_{n>N} \oint_{a(\chi)=n} h(\chi Z, \psi) h'(\chi^{-1}, \psi) \, d\chi$$

converges absolutely with sum

$$aa'\omega(-1)(1-q^{-1})^2 (Z^2\omega\omega')^{-\mathrm{ord}\,\psi} \frac{(Z^2\omega\omega')^{-N-1} q^{N+1}}{1 - qZ^{-2}(\omega\omega')^{-1}} .$$

Then we define the number $\langle h | h' \rangle_\psi$ as the finite part of the integral

$$\oint_{\mathscr{A}(F^\times)} h(\chi, \psi) h'(\chi^{-1}, \psi) \, d\chi ,$$

where a simple positive contour around the origin is taken in each component of $\mathscr{A}(F^\times)$, containing the poles of $\chi \mapsto h'(\chi^{-1}, \psi)$ but not those of $\chi \mapsto h(\chi, \psi)$. This means that $\langle h | h' \rangle_\psi$ is given for N large enough by

$$\langle h | h' \rangle_\psi = \oint_{a(\chi) \le N} h(\chi, \psi) h'(\chi^{-1}, \psi) \, d\chi + \begin{cases} 0 & \text{if } \omega\omega' \text{ ramifies}, \\ aa'\omega(-1)(1-q^{-1})^2 \dfrac{(\omega\omega')^{-N-\mathrm{ord}\,\psi}}{\omega\omega' - q} q^{N+1} & \text{otherwise}. \end{cases}$$

It is finite unless $\omega\omega' = q$. In the case $h'(\chi^{-1}, \psi) = bh(\chi, \psi)^{-1}$, we have $aa'\omega(-1) = b$, $\omega' = \omega^{-1}$, so $\omega\omega'$ is not q and

$$\langle h | h' \rangle_\psi = b \oint_{a(\chi) \le N} d\chi + b(1-q^{-1})^2 \frac{q^{N+1}}{1-q} ;$$

since $\oint_{a(\chi) \le N} d\chi$ is the measure of units t satisfying $\mathrm{ord}(t-1) \ge N$, that is,

$$q^N, (1-q^{-1}) = -(1-q^{-1})^2 \frac{q^{N+1}}{1-q} ,$$

we have shown that $\langle h | h' \rangle_\psi = 0$ in this case.

We shall write

$$\langle h | h' \rangle_\psi = \oint_{\mathscr{A}(F^\times)} h(\chi, \psi) h'(\chi^{-1}, \psi) \, d\chi .$$

An example of functions h, h' satisfying (a) and (b) are the quadratic Gauss sums γ_θ^F and $\gamma_{\theta'}^F$.

2.4. We give an example of the pairing involving the beta function. In general, for μ, ν characters of $K^\times$, and ψ a nontrivial additive character of K, we define the beta function B^K of K by

$$B^K(\mu, \nu) = \Gamma^K(\mu, \psi)\Gamma^K(\nu, \psi)/\Gamma^K(\mu\nu, \psi), \tag{2.4.1}$$

which is independent of the choice of ψ. We have used the traditional notation B^K.

Proposition. *Let K be a separable quadratic extension of F, and $\theta, \theta' \in \mathscr{A}(K^\times)$. Assume $|\theta\theta'| > q_K^{-1/2}$, and $\theta\theta' \neq 1$ if θ and θ' are liftings of characters of $F^\times$. Then*

$$\int_{\mathscr{A}(F^\times)} B^K(\theta\chi \circ N, \theta'\chi^{-1} \circ N)\, d\chi = \int_{K_T} (\theta\overline{\theta}')(y) d^\bullet y, \tag{2.4.2}$$

where the left-hand side is defined from the pairing in §2.3, *and $d^\bullet y$ has been defined in* §2.1.

Proof. The condition $|\theta\theta'| > q_K^{-1/2}$ assures the convergence of both integrals. As they are analytic in this domain, we prove the identity under the conditions $|\theta| < 1$, $|\theta'| < 1$, $|\theta\theta'| > q_K^{-1/2}$. Let m be a positive integer and choose a positive number R_m satisfying

$$\Gamma^K(\theta\chi \circ N, \psi \circ T) = \int_{|Nx| \le R_m} \theta(x)\chi(Nx)\psi(Tx) d^\times x,$$

and

$$\Gamma^K(\theta'\chi^{-1} \circ N, \psi \circ T) = \int_{|Ny| \le R_m} \theta'(y)\chi^{-1}(Ny)\psi(Ty) d^\times y$$

for all characters $\chi \in \mathscr{A}(F^\times)$ with conductor $a(\chi) \le m$. The orthogonal in $F^\times$ of this subgroup of $\mathscr{A}(F^\times)$ is $1 + \mathscr{P}^m$, where $\mathscr{P}$ is the valuation ideal of F. So $\Gamma^K(\theta\theta', \psi \circ T)$ times the left-hand side of (2.4.2) is the limit as m tends to infinity of

$$\oint_{a(\chi) \le m} \left(\int_{|Nx| \le R_m, |Ny| \le R_m} \theta(x)\theta'(y)\psi(Tx + Ty)\chi(Nx/Ny) d^\times x\, d^\times y \right) d\chi,$$

which is equal to

$$|\mathscr{O}^\times/(1 + \mathscr{P}^m)| \int_{|Nx| \le R_m, |Ny| \le R_m, N(y/x) \in 1 + \mathscr{P}^m} \theta(x)\theta'(y)\psi(Tx + Ty)\, d^\times x\, d^\times y$$

$$= |\mathscr{O}^\times/(1 + \mathscr{P}^m)| \int_{Nw \in 1 + \mathscr{P}^m, |Nx| \le R_m |N(1+w)|} (\theta\theta')(x)\psi(Tx)\theta'(w) \times (\theta\theta')(1 + w)^{-1}\, d^\times x\, d^\times w,$$

by the change of variables $(x, y) \mapsto (x(1+w)^{-1}, xw(1+w)^{-1})$. For m large enough, the subgroup $1 + \mathscr{P}^m$ is the image by N of some subgroup $1 + \mathscr{P}_K^{m'}$ such that $w \mapsto \theta'(w)(\theta\theta')(1+w)^{-1}$ on $N^{-1}(1+\mathscr{P}^m) = K_1(1+\mathscr{P}_K^{m'})$ is constant

$\bmod(1+\mathscr{P}_K^{m'})$; then, because $|\mathscr{O}^\times/(1+\mathscr{P}^m)|\int_{1+\mathscr{P}^m} d^\times t = 1$, our expression is, with the normalized Haar measure $d^\times w$ on K_1 ,

$$e\int_{x\in K^\times,\, w\in K_1,\, |Nx|\le R_m|N(1+w)|}(\theta\theta')(x)\psi(Tx)\theta'(w)(\theta\theta')(1+w)^{-1}\,d^\times x\,d^\times w\,.$$

We choose now a positive number r for which

$$\Gamma^K(\theta\theta',\,\psi\circ T)=\int_{|Nx|\le r}(\theta\theta')(x)\psi(Tx)\,d^\times x\,.$$

Then, for $|N(1+w)|>r/R_m$, the ball $|Nx|\le R_m|N(1+w)|$ contains the ball $|Nx|\le r$. We write our expression as

$$e\Gamma^K(\theta\theta',\,\psi\circ T)\int_{w\in K_1,\,|N(1+w)|>r/R_m}\theta'(w)(\theta\theta')(1+w)^{-1}\,d^\times w$$

$$+e\int_{\substack{x\in K^\times,\,w\in K_1\\ |Nx|\le R_m|N(1+w)|,\,|N(1+w)|\le r/R_m}}(\theta\theta')(x)\psi(Tx)\theta'(w)\times(\theta\theta')(1+w)^{-1}\,d^\times x\,d^\times w\,.$$

By Cayley transform $y=(1+w)^{-1}$, $\theta'(w)(\theta\theta')(1+w)$ becomes $(\theta\overline{\theta}')(y)=\theta(y)\theta'(\overline{y})$, and the first term is

$$\Gamma^K(\theta\theta',\,\psi\circ T)\int_{K_T,\,|Ny|<R_m/r}(\theta\overline{\theta}')(y)d^\bullet y\,.$$

The assumption $|\theta\theta'|>q_K^{-1/2}$ implies that this integral has a limit when m , hence R_m , goes to infinity, by Lemma 2.2. We prove now that the second term goes to 0. Let σ be the real number such that $|\theta\theta'(x)|=|Nx|^\sigma$. By assumption, $\sigma<1/2$. The second term now reads

$$\int_{\substack{x\in K^\times,\,y\in K_T\\ |N(xy)|\le R_m,\,|Ny|\ge R_m/r}}(\theta\theta')(x)(\theta\overline{\theta}')(y)\psi(Tx)\,d^\times x\,d^\bullet y$$

and, in absolute value, is dominated by

$$\int_{\substack{x\in K^\times,\,y\in K_T\\ |N(xy)|\le R_m,\,|Ny|\ge R_m/r}}|N(xy)|^\sigma\,d^\times x\,d^\bullet y$$

$$=\int_{x\in K^\times,\,|Nx|\le R_m}|Nx|^\sigma\,d^\times x\int_{y\in K_T,\,|Ny|\ge R_m/r}d^\bullet y\,.$$

In the right-hand side, the first integral is $O(R_m^\sigma)$, the second is $O(R_m^{-1/2})$ as seen in the proof of lemma. So, the second term is $O(R_m^{\sigma-1/2})$, and goes to 0 when m goes to infinity. This achieves the proof of the proposition.

2.5. Given two quadratic étale F-algebras K and K' , and $B=K\otimes_F K'$, we denote by B_* the subgroup of $K^\times\times K'^\times$ consisting of elements $(x\,,\,x')$ such that $N_{K/F}x = N_{K'/F}x'$. It is a closed subgroup of $K^\times \times K'^\times$ and we have a homomorphism from $B^\times$ to B_* given by

$$x \mapsto (N_{B/K}x, N_{B/K'}x) \tag{2.5.1}$$

with kernel $K''_1 = \operatorname{Ker} N_{K''/F}$, where K'' denotes the third quadratic sub-F-algebra in B. As $\mathscr{O}_K^\times \times \mathscr{O}_{K'}^\times$ is the maximal compact subgroup of $K^\times \times K'^\times$, its intersection $\mathscr{O}_{B*}^\times$ with B_* is the maximal compact subgroup of B_*, and it is open in B_*. The Haar measure $d_{B_*}^\times$ gives to $\mathscr{O}_{B_*}^\times$ the volume 1. The group B_* appears also as the orthogonal in $K^\times \times K'^\times$ of the group $\mathscr{A}(F^\times)$ embedded in $\mathscr{A}(K^\times \times K'^\times)$ by $\chi \mapsto (\chi \circ N_{K/F}, \chi^{-1} \circ N_{K'/F})$. By Poisson summation formula [W1], this implies that there is a number $c_1 > 0$ such that, for h in the Schwartz-Bruhat space $\mathscr{S}(K^\times \times K'^\times)$ one has

$$\begin{aligned} &\int_{\mathscr{A}(F^\times)} \left(\int_{K^\times \times K'^\times} h(x, x')\chi(N_{K/F}x/N_{K'/F}x')\, d_K^\times x\, d_{K'}^\times x' \right) d\chi \\ &\quad = c_1 \int_{B*} h(x, x') d_{B_*}^\times (x, x'). \end{aligned} \tag{2.5.2}$$

Lemma. *Assume that K and K' are not isomorphic. Then*

(a) *the image of $B^\times$ in B_* by (2.5.1) is an index 2 subgroup;*

(b) *the restriction of (2.5.1) to $\mathscr{O}_B^\times$ has image in $\mathscr{O}_{B_*}^\times$ a subgroup of index $e_{K''/F}$ if K or K' splits over F, and of index 2 otherwise;*

(c) *$c_1 = 1$ if K or K' splits over F, otherwise $c_1 = f_{K''/F}$, the modular degree of K'' over F.*

Proof. (1) Assume first $K' = F \times F$. Then B_* is the set of $(w, (u, v)) \in K^\times \times (F \times F)^\times$ such that $w\overline{w} = uv$. The algebra B is $K \times K$ and (2.5.1) is $(x, y) \mapsto (xy, (x\overline{x}, y\overline{y}))$. We check now that the image of $B^\times$ is the kernel of the map $(w, (u, v)) \mapsto \eta_{K/F}(u)$ on B_*: if $(w, (u, v))$ lies in B_* and u is a norm $x\overline{x}$ from $K^\times$, so is $v = w\overline{w}/u$. Put $y = wx^{-1}$, then $v = y\overline{y}$ and $w = xy$. Thus $(w, (u, v)) = (xy, (x\overline{x}, y\overline{y}))$ lies in the image of $B^\times$. This gives an isomorphism between the cokernel of the map (2.5.1) and the cokernel of $\eta_{K/F}$, and proves (a) in our case. For (b), we observe that the inverse image of $\mathscr{O}_{B_*}^\times$ in $B^\times$ is the group $\mathscr{O}_B^\times$ of units of B, and that the cokernel of (2.5.1) restricted to $\mathscr{O}_B^\times$ is isomorphic to $\mathscr{O}_F^\times / N_{K/F}\mathscr{O}_K^\times$, which has order equal to the ramification index $e_{K/F}$ of K over F. For (c), we apply (2.5.2) to the characteristic function of $\mathscr{O}_K^\times \times \mathscr{O}_{K'}^\times$: the integral

$$\int_{\mathscr{O}_K^\times \times \mathscr{O}_F^\times \times \mathscr{O}_F^\times} \chi(w\overline{w})\chi^{-1}(uv)\, d_K^\times w\, d_F^\times u\, d_F^\times v$$

is 0 unless χ is unramified, and then the left-hand side of (2.5.2) is 1, as is the right-hand side when $c_1 = 1$. This proves the lemma when K or K' splits.

(2) Assume now that K and K' are not isomorphic and nonsplit. Let $(u, v) \in B_*$; then the equality $N_{K/F}u = N_{K'/F}v$ shows that this element of $F^\times$ lies in $\operatorname{Im} N_{K/F} \cap \operatorname{Im} N_{K'/F} = \operatorname{Im} N_{B/F}$. Let $z \in B^\times$ with $N_{B/F}z$ equal to this common norm. Its image by (2.5.1) differs from (u, v) by some element of $K_1 \times K'_1$, the product of $\operatorname{Ker} N_{K/F}$ by $\operatorname{Ker} N_{K'/F}$. As the inverse image of $K_1 \times K'_1$ in $B^\times$ by (2.5.1) is $B_1 = \operatorname{Ker} N_{B/F}$, we have shown that the orbits of $B^\times$ acting on B_* through (2.5.1) are the same as the orbits of B_1 acting on $K_1 \times K'_1$. As any element of K_1 lies in $\operatorname{Im} N_{B/K}$, so is in $N_{B/K}B_1$, the number of these orbits is $[K'_1 : N_{B/K'}B_K]$ where $B_K = \operatorname{Ker} N_{B/K}$. We prove now that $N_{B/K'}B_K$ has index 2 in K'_1. For that, we use the long exact sequence for the cohomology of the group $\operatorname{Gal} B/K''$ acting on the exact sequence

$$1 \to K^\times \to B^\times \to B_K \to 1$$

of subgroups of $B^\times$ by its Galois action composed with inversion:

$$1 \to K_1 \to B_{K''} \to K'_1 \to F^\times/N_{K/F}K^\times \xrightarrow{\alpha} K''^\times/N_{B/K''}B^\times$$
$$\to K''_1/N_{B/K''}K^\times \to 0 \to \cdots .$$

As any element of $F^\times$ is a norm from B to K'', the arrow α is 0. The image of $B_{K''}$ in K'_1 is $N_{B/K'}B_K$, and the factor group is isomorphic to $F^\times/N_{K/F}K^\times$, which has order 2. This proves the claim, which gives part (a) of the lemma and also part (b) since B_1 consists of units. For (c), we use again the characteristic function of $\mathcal{O}_K^\times \times \mathcal{O}_{K'}^\times$; this shows that c_1 is the index in $\mathcal{O}_F^\times$ of the subgroup generated by $N_{K/F}\mathcal{O}_F^\times$ and $N_{K'/F}\mathcal{O}_{K'}^\times$. As $N_{K/F}\mathcal{O}_K^\times = \mathcal{O}_F^\times$ if K is unramified over F, and $[\mathcal{O}_F^\times : N_{K/F}\mathcal{O}_K^\times] = 2$ otherwise, we see that $c_1 = 1$ unless $N_{K/F}\mathcal{O}_K^\times = N_{K'/F}\mathcal{O}_{K'}^\times$; in this case, the units of F are either in $N_{K/F}K^\times \cap N_{K'/F}K'^\times$ or in the complement of $N_{K/F}K^\times \cup N_{K'/F}K'^\times$, that is, are all in $N_{K''/F}K''^\times$ since $F^\times$ is the union $\operatorname{Im} N_{K/F} \cup \operatorname{Im} N_{K'/F} \cup \operatorname{Im} N_{K''/F}$; but then, K'' is unramified, so $f_{K''/F} = 2$. This gives the proof of (c).

3. Main results

3.1. Theorem 1. *Let K and K' be two quadratic étale F-algebras, and let B be their composite algebra $K \otimes_F K'$. Let θ, θ' be characters of $K^\times$ and $K'^\times$ respectively. Denote by $\omega\eta_{K/F}$, $\omega'\eta_{K'/F}$ the restrictions of θ, θ' to $F^\times$, and by $\theta \times \theta'$ the character $(\theta \circ N_{B/K})(\theta' \circ N_{B/K'})$ of $B^\times$. Then we have*

$$\langle \gamma_\theta^F | \gamma_{\theta'}^F \rangle_\psi = \gamma_{\theta\times\theta'}^F(q^{-1/2}, \psi)/\Gamma(\omega\omega' q^{-2}, \psi). \tag{3.1.1}$$

Proof. (a) If K and K' are both split, we see them in $B = F^4$ as the fixed points of the involutions

$$(t, u, v, w) \mapsto (v, w, t, u)$$

and

$$(t, u, v, w) \mapsto (w, v, u, t)$$

respectively. For $\theta = \mu \otimes \nu$, $\theta' = \mu' \otimes \nu'$, we have

$$\theta \times \theta' = (\mu\mu') \otimes (\nu\nu') \otimes (\mu\nu') \otimes (\nu\mu').$$

In this case, formula (3.1.1) has been proved in [L and GL1].

(b) If K and K' are isomorphic fields, we see them as fixed points in $B = K \times K$ under the involutions $(x, y) \mapsto (y, x)$ and $(x, y) \mapsto (\bar{y}, \bar{x})$ respectively, with $^-$ the conjugation of K over F. Then, $\theta \times \theta'$ is the character $(x, y) \mapsto (\theta\theta')(x)(\theta\bar{\theta}')(y)$, and

$$\gamma^F_{\theta\times\theta'}(q^{-1/2}, \psi) = \lambda_{B/K}(\psi)^2 \gamma^K(\theta\theta' q_K^{-1/2}, \psi \circ T)\gamma^K(\theta\bar{\theta}' q_K^{-1/2}, \psi \circ T)$$

with $T = T_{K/F}$. Multiply both θ and θ' by $q_K^{1/2}$, then $\omega\omega'$ is multiplied by q^2 and (3.1.1) takes the form

$$(3.1.2) \qquad \oint_{\mathscr{A}(F^\times)} B^K(\theta\chi \circ N, \theta'\chi^{-1} \circ N)\, d\chi = \Gamma^K(\theta\bar{\theta}', \psi \circ T)/\Gamma(\omega\omega', \psi),$$

with $N = N_{K/F}$ and B^K the beta function (2.4.1) for K. Formula (3.1.2) is the proposition of §2.4 combined with the lemma of §2.2.

(c) The case of K and K' nonisomorphic remains. As they play the same role in the statement, we assume that K is a field. Then the third quadratic étale F-algebra K'' in B defined by K and K' is also a field, satisfying

$$\eta_{K/F}\eta_{K'/F}\eta_{K''/F} = 1.$$

We choose two coset representatives, say, a_+ and a_- of $N_{K''/F}(K''^\times)$ in $F^\times$. Using the complement formula in §1.1 for Γ, we rewrite (3.1.1) as

$$(3.1.3) \qquad \langle \gamma^F_\theta | \gamma^F_{\theta'} \rangle_\psi = |\psi|(\omega\omega')(-1)L_F^{-2}\Gamma(\omega^{-1}\omega'^{-1}q, \psi^{-1})\gamma^F_{\theta\times\theta'}(q^{-1/2}, \psi).$$

Since both sides of (3.1.3) are meromorphic functions in θ, θ', it suffices to prove the identity for $q < |\omega\omega'| < q^2$, and we shall so assume. Our strategy is to express the right-hand side of (3.1.3) as an integral over the subgroup B_* of $K^\times \times K'^\times$, the orthogonal of the group $\mathscr{A}(F^\times)$ embedded in $\mathscr{A}(K^\times \times K'^\times)$ by $\chi \mapsto (\chi \circ N_{K/F}, \chi^{-1} \circ N_{K'/F})$; a suitable form of Poisson summation formula expresses then the right-hand side as a contour integral over $\mathscr{A}(F^\times)$, which is the left-hand side of (3.1.3).

The condition $|\omega\omega'| > q$ implies the integrability near 0 for $d^\times t$ of the character $\omega^{-1}\omega'^{-1}q$. Thus, for R large, one has

$$\Gamma(\omega^{-1}\omega'^{-1}q, \psi^{-1}) = \int_{|t|\le R} (\omega\omega')(t)^{-1}\psi(-t)|t|^{-1}\, d^\times t,$$

and

$$\Gamma(\omega^{-1}\omega'^{-1}\eta'' q, \psi^{-1}) = \int_{|t|\le R} (\omega\omega')(t)^{-1}\eta''(t)\psi(-t)(t)^{-1}\, d^\times t,$$

with $\eta'' = \eta_{K''/F}$, so

$$\Gamma(\omega^{-1}\omega'^{-1}q, \psi^{-1}) = \Gamma_+ + \Gamma_-,$$

where $\Gamma_\pm$ correspond to the intersection over those t in the ball $|t| \le R$ which satisfy $t \in a_\pm N''(K''^\times)$, with $N'' = N_{K''/F}$. We express $\Gamma_\pm$ as integrals over K'':

(3.1.4)

$$\begin{aligned}\Gamma_\pm &= \Gamma_{\pm, R}\\ &= c^{-1}|a_\pm|^{-1}\int_{|N''w|\le R}(\omega\omega')(a_\pm N''w)^{-1}\psi(-a_\pm N''w)|N''w|^{-2}d_{K''/F}w,\end{aligned}$$

with c being the measure of $K_1'' = \operatorname{Ker} N''$ under $d_{K''/F}w/d^\times t$.

The condition $|\omega\omega'| < q^2$ assures that the character $\beta = \theta \times \theta'$ of $B^\times$ is integrable near 0 with respect to the measure $d_{B,\psi}$; so, for S large, we have

$$\gamma_\beta^B(q_B^{-1/2}, \psi \circ T_{B/F}) = \int_{|z|_B \le S}\beta(z)\psi\circ T_{B/F}(z)d_{B,\psi}z,$$

where $|z|_B = \max(|Nx|, |Ny|)$ if K' is split and z in B correspond to (x, y) in $K \times K$, and $N = N_{K/F}$.

Our first step is to express the product of the two gamma functions in the right-hand side of (3.1.3) as follows:

Lemma.

$$\Gamma(\omega^{-1}\omega'^{-1}q, \psi)\gamma_\beta^B(q_B^{-1/2}, \psi\circ T_{B/F}) = |\psi|^{-1}\lambda''(\psi)^{-1}c^{-1}c'\lim_{Q\to\infty}A_Q$$

where $\lambda''(\psi) = \lambda_{K''/F}(\psi)$ *and*

$$A_Q = \int_{(x, x')\in B_*, |Nx|\le Q}(\theta\psi\circ T)(x)(\theta'\psi\circ T')(x')d^*_{B_*,\psi}(x, x'),$$

with $T = T_{K/F}$, $T' = T_{K'/F}$, c' *is the index of the image of* $\mathscr{O}_B^\times$ *in* $\mathscr{O}_{B_*}^\times$ *by* (2.5.1) *and* $d^*_{B_*,\psi}(x, x') = |Nx|d_{B_*,\psi}(x, x')$.

Proof of the lemma. Choose S large as above, then

$$\begin{aligned}&c\Gamma(\omega^{-1}\omega'^{-1}q, \psi^{-1})\gamma^B(\beta q_B^{-1/2}, \psi\circ T_{B/F})\\ &\quad= c(\Gamma_+ + \Gamma_-)\gamma^B(\beta q_B^{-1/2}, \psi\circ T_{B/F})\\ &\quad= \lim_{r\to 0} c(\Gamma_+ - \Gamma_{+, r|a_+|^{-2}} + \Gamma_- - \Gamma_{-, r|a_-|^{-2}})\gamma^B(\beta q_B^{-1/2}, \psi\circ T_{B/F})\\ &\quad= \lim_{r\to 0}(I(r, S, a_+) + I(r, S, a_-)),\end{aligned}$$

where, for $a = a_+$ or a_- and R large as before, we define

$$\begin{aligned}I(r, S, a) = |a|^{-1}\int_{|z|_B\le S, r|a|^{-2}\le|N''w|\le R}&(\omega\omega')(aN''w)^{-1}\psi(-aN''w)\beta(z)\\ &\times\psi\circ T_{B/F}(z)|N''w|^{-2}d_{K''/F}w\,d_{B,\psi}z.\end{aligned}$$

Note that for $w \in K''^{\times}$, we have

$$\beta(w) = \theta(N_{B/K}w)\theta'(N_{B/K'}w) = \theta(N''w)\theta'(N''w) = (\omega\omega'\eta\eta')(N''w) = (\omega\omega')(N''w),$$

with $\eta = \eta_{K/F}$, $\eta' = \eta_{K'/F}$ so that $\eta'' = \eta\eta'$. Therefore, $I(r, S, a)$ can be simplified as

$$|a|^{-1}(\omega\omega')(a)^{-1}\int_{|z|_B\leq S,\, r|a|^{-2}\leq|N''w|\leq R}\beta(w^{-1}z)\psi(-aN''w) \times \psi\circ T_{B/F}(z)|N''w|^{-2}d_{K''/F}w\,d_{B,\psi}z$$

$$= |a|^{-1}(\omega\omega')(a)^{-1}\int_{|z|_B\leq Sr^{-1}|a|^2,\, r|a|^{-2}\leq|N''w|\leq R}\beta(z)\psi(-aN''w) \times \psi\circ T_{B/F}(wz)d_{K''/F}w\,d_{B,\psi}z$$

by the change of variable $z\mapsto wz$. For a given S, the assumption $q < |\omega\omega'| < q^2$ implies that the integral

$$\int_{|z|_B\leq Sr^{-1},\, |N''w|<r}|\beta(z)|d_{K''/F}w\,d_{B,\psi}z$$

is majorized by a constant multiple of $r\int_{|z|_B\leq Sr^{-1}}|\beta(z)|d_Bz$, which is $O(r^{\alpha})$ with $\alpha = |\beta|^{(\log q_B)^{-1}} > 0$. This shows that the difference

$$I(r, S, a) - I(0, Sr^{-1}|a|^2, a)$$

tends to zero as r does. Hence we have

$$c\Gamma(\omega^{-1}\omega'^{-1}q, \psi^{-1})\gamma^B(\beta q_B^{-1/2}, \psi\circ T_{B/F}) = \lim_{Q\to\infty}(I(Q|a_+|^2, a_+)+I(Q|a_-|^2, a_-))$$

where for R and Q large $I(Q, a) = I(0, Q, a)$, that is,

$$I(Q, a) = |a|^{-1}(\omega\omega')(a)^{-1}\int_{|z|_B\leq Q,\, |N''w|\leq R}\beta(z)\psi(-aN''w) \times \psi\circ T_{B/F}(wz)d_{K''F}w\,d_{B,\psi}z.$$

For fixed a and for R large enough, the integral against w has been computed in the lemma of §1.5. It is 0 unless $|N''\circ T_{B/K''}z| \leq R$; take R large enough, the condition $|z|_B \leq Q$, or equivalently $|N_{B/K''}z|_{K''} \leq R$, implies $|T_{B/K''}z|_{K''} \leq R$, and this integral against w is, for $|z|_B \leq Q$, equal to

$$|\psi|^{-1}|a|^{-1}\eta''(a)\lambda''(\psi)^{-1}\psi(a^{-1}N''\circ T_{B/K''}z).$$

Noting

$$(\omega\omega')(a)^{-1} = \theta(a)^{-1}\theta'(a)^{-1}\eta(a)\eta'(a) = \theta(a)^{-1}\theta'(a)^{-1}\eta''(a)$$

and

$$N''\circ T_{B/K''} = T\circ N_{B/K} + T'\circ N_{B/K'},$$

we arrive at

$$I(Q|a|^2, a) = |\psi|^{-1}\lambda''(\psi)^{-1}|a|^{-2}\int_{|z|_B\leq Q|a|^2}(\theta\psi\circ T)(a^{-1}N_{B/K}z)$$
$$\times(\theta\psi\circ T')(a^{-1}N_{B/K'}z)\,d_{B/\psi'}z\,.$$

From the lemma in §2.5, the image of $B^\times$ in $K^\times\times K'^\times$ by the map $(N_{B/K}, N_{B/K'})$ is a subgroup B_*^+ of index 2 of B_* with coset representatives (a_+^{-1}, a_+^{-1}) and (a_-^{-1}, a_-^{-1}). We express $I(Q|a|^2, a)$ as an integral over B_*^+:

$$I(Q|a|^2, a) = |\psi|^{-1}\lambda''(\psi)^{-1}c'\int_{(x,x')\in B_*^+, |Nx|\leq Q|a|^2}|a|^{-2}(\theta\psi\circ T)(a^{-1}x)$$
$$\times(\theta'\psi\circ T')(a^{-1}x')d^*_{B_*,\psi}(x, x')$$
$$= |\psi|^{-1}\lambda''(\psi)^{-1}c'\int_{(x,x')\in a^{-1}B_*^+, |Nx|\leq Q}(\theta\psi\circ T)(x)(\theta'\psi\circ T')(x')d^*_{B_*,\psi}(x, x')$$

by the change of variables $(x, x')\mapsto(ax, ax')$. Here c' is the index of the maximal compact subgroup of B_*^+ in the maximal compact subgroup of B_*. Adding $I(Q|a_+|^2, a_+)$ and $I(Q|a_-|^2, a_-)$ leads to

$$|\psi|^{-1}\lambda''(\psi)^{-1}c'\int_{(x,x')\in B_*, |Nx|\leq Q}(\theta\psi\circ T)(x)(\theta'\psi\circ T')(x')d^*_{B_*,\psi}(x, x')$$
$$= |\psi|^{-1}\lambda''(\psi)^{-1}A_Q\,.$$

This proves the lemma.

Recall that $\lambda_{B/F} = \lambda(\psi)\lambda'(\psi)\lambda''(\psi)$, where $\lambda(\psi) = \lambda_{K/F}(\psi)$, $\lambda'(\psi) = \lambda_{K'/F}(\psi)$, $\lambda''(\psi) = \lambda_{K''/F}(\psi)$, as seen in §1. The lemma gives the right-hand side of (3.1.3) as

(3.1.5)
$$L_F^{-2}\lambda(\psi)\lambda'(\psi)c^{-1}c'\lim_{Q\to\infty}\int_{(x,x')\in B_*, |Nx|\leq Q}(\theta\psi\circ T)(x)(\theta'\psi\circ T')(x')d^*_{B_*}(x, x')\,.$$

On the other hand, for $|\omega\omega'| < q^2$, the left-hand side of (3.1.3) is given by a convergent integral

$$\lambda(\psi)\lambda'(\psi)\oint_{\mathscr{A}(F^\times)}\gamma^K(\theta\chi\circ N, \psi\circ T)\gamma^{K'}(\theta'\chi^{-1}\circ N', \psi\circ T')\,d\chi\,.$$

We assume now $|\omega|$ and $|\omega'| < q$, which is compatible with the preceding assumption $q < |\omega\omega'| < q^2$. For χ unitary with $a(\chi)\leq M$, the two gamma terms $\gamma^K(\theta\chi\circ N, \psi\circ T)$, $\gamma^{K'}(\theta'\chi^{-1}\circ N', \psi\circ T')$ above are given by integrals on sufficiently large compact subsets of K and K', depending on M, and independent of θ, θ' running through given compact subsets of $\mathscr{A}(K^\times)$, $\mathscr{A}(K'^\times)$:

$$\gamma^K(\theta\chi\circ N, \psi\circ T) = \int_{|Nx|\leq Q(M)}(\theta\psi\circ T)(x)\chi\circ N(x)d^*_{K,\psi}x\,,$$

$$\gamma^{K'}(\theta'\chi^{-1}\circ N', \psi\circ T) = \int_{|x'|_{K'}\le Q(M)} (\theta'\psi\circ T')(x')\chi^{-1}\circ N'(x')d^*_{K',\psi}x'.$$

Using (3.1.5), we see that identity (3.1.3) is equivalent to

(3.1.6)

$$\begin{cases} \lim\limits_{M\to\infty} \chi_{a(\chi)\le M}\left(\int_{|Nx|\le Q(M),\,|x'|_{K'}\le Q(M)} f(x, x')x(Nx/N'x')d^*_{K,\psi}xd^*_{K',\psi}x'\right) d\chi \\ = c^{-1}c'L_F^{-2}\lim\limits_{Q\to\infty}\int_{(x,x')\in B_*,\,|Nx|\le Q} f(x, x')d^*_{B_*,\psi}(x, x'), \end{cases}$$

where $f(x, x') = (\theta\psi\circ T)(x)(\theta'\psi\circ T')(x')$ for $x\in K^\times$, $x'\in K'^\times$. Recall that the measures $d^*_{K,\psi}$, $d^*_{K',\psi}$, $d^*_{B_*,\psi}$ are given by

$$d^*_{K,\psi}x = |\psi|D^{-1/2}L_K^{-1}|Nx|^{1/2}d^\times_K x, \qquad D = D_{K/F},$$

$$d^*_{K',\psi}x' = |\psi|D'^{-1/2}L_{K'}^{-1}|N'x'|^{1/2}d^\times_{K'}x', \qquad D' = D_{K'/F},$$

$$d^*_{B_*,\psi}(x, x') = |\psi|^2D_{B/F}^{-1/2}L_B^{-1}|Nx|^{1/2}|N'x'|^{1/2}d^\times_{B_*}(x, x'),$$

hence, by §1.6, we rewrite (3.1.6) as

(3.1.7)

$$\begin{cases} \lim\limits_{Q\to\infty} \chi_{a(\chi)\le M}\left(\int_{|Nx|\le Q(M),\,|x'|_{K'}\le Q(M)} h(x, x')\chi(Nx/N'x')^\times_K xd^\times_{K'}x'\right) dx \\ = c'e''^{-1}\lim\limits_{Q\to\infty}\int_{(x,x')\in B_*,\,|Nx|\le Q} h(x, x')d^\times_{B_*}(x, x'), \end{cases}$$

where $e'' = e_{K''/F}$, and h is now the function on $K^\times\times K'^\times$ given by

$$h(x, x') = (\theta\psi\circ T)(x)(\theta'\psi\circ T')(x')|Nx|^{1/2}|N'x'|^{1/2}.$$

We observe now that for $|\omega| < q$ and $|\omega'| < q$, the function h_M on $K^\times\times K'^\times$ equal to h on $|Nx|\le Q(M)$, $|x'|_{K'}\le Q(M)$, and 0 otherwise, lies in $\mathscr{S}(K^\times\times K'^\times)$. We apply Poisson summation formula (2.5.2) to this function h_M:

$$\oint_{\mathscr{A}(F^\times)}\left(\int_{K^\times\times K'^\times} h_M(x, x')\chi(Nx/N'x')\,d^\times_K x'\right) d\chi$$
$$= c_1\int_{B_*} h_M(x, x')\,d^\times_{B_*}(x, x').$$

On the left-hand side, only the characters χ with $a(\chi)\le M$ will contribute, so this formula is

$$\oint_{a(\chi)\le M}\left(\int_{|Nx|\le Q(M),\,|x'|_{K'}\le Q(M)} h(x, x')\chi(Nx/N'x')\,d^\times_K xd^\times_{K'}x'\right) d\chi$$
$$= c_1\int_{(x,x')\in B_*,\,|Nx|\le Q(M)} h(x, x')\,d^\times_{B_*}(x, x').$$

This clearly proves (3.1.7), and hence completes the proof of the theorem if we show $c_1 = c'e''^{-1}$; but this is proved in the lemma of §2.5.

3.2. The multiplicative formula for γ_θ^F.

Theorem 2. *Let K be a quadratic étale F-algebra, and $\theta \in \mathscr{A}(K^\times)$. If $\alpha, \beta \in \mathscr{A}(F^\times)$ are not poles of $\chi \mapsto \gamma_\theta^F(\chi, \psi)$, then*

$$\oint_{\mathscr{A}(F^\times)} \gamma_\theta^F(\chi, \psi)\Gamma(\alpha\chi^{-1}, \psi)\Gamma(\beta\chi^{-1}, \psi)\, d\chi = \Gamma((\alpha\beta\omega)^{-1}, \psi^{-1})\gamma_\theta^F(\alpha, \psi)\gamma_\theta^F(\beta, \psi) \tag{3.2.1}$$

where $\omega = \eta_{K/F}\theta|_{F^\times}$.

Proof. We apply Theorem 1 to $K' = F \times F$, $\theta'(u, v) = \alpha(u)\beta(v)|uv|^{-1/2}$. Then B is $K \times K$ and

$$(\theta \times \theta')(xy) = \theta(xy)\alpha(Nx)\beta(Ny)|N(xy)|^{-1/2},$$

with $N = N_{K/F}$. As $\lambda_{B/F} = \lambda_{K/F}(\psi)^2$, using the complement formula in §1.1 for Γ, the right-hand side of (3.1.1) is written as

$$|\psi|L_F^{-2}\Gamma((\alpha\beta\omega)^{-1}, \psi^{-1})\gamma_\theta^F(\alpha, \psi)\gamma_\theta^F(\beta, \psi).$$

On the other hand,

$$\begin{aligned}\gamma_{\theta'}^F(\chi^{-1}, \psi) &= \gamma^F(\alpha q^{1/2}\chi^{-1}, \psi)\gamma^F(\beta q^{1/2}\chi^{-1}, \psi)\\ &= (|\psi|^{1/2}L_F^{-1})^2\Gamma(\alpha\chi^{-1}, \psi)\Gamma(\beta\chi^{-1}, \psi).\end{aligned}$$

This gives (3.2.1).

3.3. Theorem 3. *Let K and K' be two nonisomorphic étale F-algebras, and let B be their tensor product over F. Let θ and θ' be two multiplicative characters of K and K' respectively. Assume that the product of the restrictions of θ and θ' to $F^\times$ is trivial. Then, for any nontrivial additive character ψ of F, one has*

$$\gamma^B(\theta \circ N_{B/K}\theta' \circ N_{B/K'}, \psi \circ T_{B/F}) = \theta(-1) = \theta'(-1).$$

Proof. Write β for the character $\theta \circ N_{B/K}\theta' \circ N_{B/K'}$ of $B^\times$. For t in $F^\times$, one has

$$\beta(t) = \theta(t^2)\theta'(t^2) = (\theta\theta')(t^2) = 1,$$

hence $|\beta| = 1$. The identity to prove is equivalent to the relation

$$\int_{B^\times} \beta(z)\hat{f}(z)d_B^*z = \theta(-1)\int_{B^\times} \beta(z)^{-1}f(z)d_B^*z \tag{3.3.1}$$

for any f in (B); the integrals are both convergent.

Denote by K'' the third quadratic F-algebra in B determined by K and K'. Since K and K' are not isomorphic, K'' is a field. By restriction to K'', the two automorphism groups of B over K and K' identify them with $\mathrm{Gal}\,K''/F$. Hence, the norm maps $N_{B/K}$ and $N_{B/K'}$ coincide on K'' with $N_{K''/F}$. As a consequence the restriction of β to the multiplicative group of

K'' is trivial. Denote by $d^*_{B,K''}$ the quotient measure of d^*_B by $d_{K''}$, so, for $f \in \mathcal{S}(B)$,

$$(3.3.2) \qquad \int_{B^\times} \beta(z)^{-1} f(z) d^*_B z = \int_{B^\times/K''^\times} \beta(z)^{-1} \left(\int_{K''} f(wz) d_{K''} w \right) d^*_{B,K''} z.$$

We apply now the Poisson formula to the closed subgroup K'' of B and to the function f

$$(3.3.3) \qquad \int_{K''} f(wz)\, d_{K''} w = |z|_B^{-1} \int_{K''_B} \hat{f}(z^{-1} w')\, d_{K''_B} w',$$

where K''_B denotes the orthogonal of K'' in B with respect to the self-duality $(z, z') \mapsto \psi \circ T_{B/F}(zz')$, that is, $K''_B = \operatorname{Ker} T_{B/K''}$, and where $d_{K''_B}$ denotes the Haar measure on K''_B associated to $d_{K''}$. For w' in K''_B, one has $N_{B/K} w' = -N_{B/K'} w'$, hence the image of K''_B under $N_{B/K}$ is contained in $K \cap K' = F$. So if $w' \in K''_B$ is not 0, one has

$$(3.3.4) \qquad \beta(w') = (\theta \circ N_{B/K} w')(\theta' \circ N_{B/K'} w') = \theta(-1) = \theta'(-1).$$

Choose a nonzero element, say s, in K''_B. We rewrite the right-hand side of (3.3.1) using (3.3.2)–(3.3.4) as

$$(3.3.5) \qquad \int_{B^\times/K''^\times} \beta(sz^{-1}) \left(\int_{K''_B} \hat{f}(z^{-1} w')\, d_{K''_B} w' \right) |z|_B^{-1}\, d^*_{B,K''} z.$$

We observe now that the measure $|z|_B^{-1} d^*_{B,K''} z$ is $d^*_{B,K''} z^{-1}$. We change the variables $w' = sw$, $z \mapsto sz^{-1}$ in (3.3.5) to get

$$\int_{B^\times/K''^\times} \beta(z) \left(\int_{K''} \hat{f}(zw) d_{K''} w \right) d^*_{B,K''} z$$

which is $\int_{B^\times} \beta(z) \hat{f}(z) d^*_B z$, the left-hand side of (3.3.1). This proves the theorem.

Corollary. *With the same assumptions on K and K', if now the product of the restrictions to $F^\times$ of θ and θ' is the character $q\colon t \mapsto |t|^{-1}$, then*

$$(3.3.6) \qquad \langle \gamma^F_\theta | \gamma^F_{\theta'} \rangle_\psi = \frac{\lambda_{B/K} \omega(-1)}{\Gamma(\eta_{K/F} \eta_{K'/F} q^{-1}, \psi)},$$

where $\omega(-1) = \theta(-1) \eta_{K/F}(-1)$.

Proof. This follows immediately from Theorems 1 and 3, due to the definition of $\lambda_{B/K}$ given in §1.6.

3.4. Theorem 4. *Let K and K' be two quadratic étale F-algebras, and $\theta \in \mathcal{A}(K^\times)$, $\theta' \in \mathcal{A}(K'^\times)$. Then $\gamma^F_\theta = \gamma^F_{\theta'}$ if and only if one of the following holds:*

(1) *for K' isomorphic to K, then θ' corresponds to θ or to $\overline{\theta}$ by such an isomorphism;*

(2) *for K' not isomorphic to K, let $B = K \otimes K'$; then θ and θ' have the same lifts to $B^\times$ and different restrictions to $F^\times$.*

Proof. If K and K' are split, then $\gamma_\theta^F(\chi, \psi) = \gamma^F(\mu\chi, \psi)\gamma^F(\nu\chi, \psi)$ if $\theta = \mu\otimes\nu$, and $\{\mu, \nu\}$ is determined by the zeros (or the poles), with multiplicities, of the function of γ_θ^F. This establishes the theorem in this case.

Assume now K is a field. If $\gamma_\theta^F = \gamma_{\theta'}^F$, the deep twist property shows that $\omega = \omega'$, hence $\theta'|_{F^\times} = \eta_{K/F}\eta_{K'/F}\theta|_{F^\times}$.

(a) If K' is not isomorphic to K, then $\eta_{K/F}\eta_{K'/F}$ is nontrivial, since the third quadratic subalgebra of B is a field.

(b) The equality $\gamma_\theta^F = \gamma_{\theta'}^F$ implies $\gamma_{\theta^{-1}}^F(\chi, \psi) = \omega(-1)\gamma_{\theta'}^F(\chi^{-1}, \psi)^{-1}$, so, as seen in §2.3, $\langle\gamma_{\theta^{-1}}^F|\gamma_{\theta'}^F\rangle_\psi = 0$. By Theorem 1, we then have

$$\gamma^B(\theta^{-1}\circ N_{B/K}\theta'\circ N_{B/K'}, q_B^{-1/2}, \psi\circ T_{B/F}) = 0.$$

(c) If K and K' are isomorphic fields, then B appears as $K\times K$ and γ^B as product of two γ^K's, so

$$\gamma^K(\theta^{-1}\theta' q_K^{-1/2}, \psi\circ T_{K/F})\gamma^K(\theta^{-1}\overline{\theta}' q_K^{-1/2}, \psi\circ T_{K/F}) = 0$$

which means $\theta' = \theta$ or $\overline{\theta}$.

(d) If now B is a field, then $\theta^{-1}\circ N_{B/K}\theta'\circ N_{B/K'} = 1$, and θ and θ' have the same lift to $B^\times$.

(e) If K is a field and K' is $F\times F$, then $\theta' = \mu\otimes\nu$ with $\mu, \nu \in \mathscr{A}(F^\times)$ and B is $K\times K$; then

$$\gamma^K(\mu\circ N_{K/F}\theta^{-1}q_K^{-1/2}, \psi\circ T_{K/F})\gamma^K(\nu\circ N_{K/F}\theta^{-1}q_K^{-1/2}, \psi\circ T_{K/F}) = 0,$$

so θ is either $\mu\circ N_{K/F}$ or $\nu\circ N_{K/F}$. By the Davenport-Hasse theorem we have $\gamma_\theta^F(\chi, \psi) = \gamma^F(\mu\chi, \psi)\gamma^F(\mu\eta_{K/F}\chi, \psi)$ or $\gamma^F(\nu\chi, \psi)\gamma^F(\nu\eta_{K/F}\chi, \psi)$. As $\gamma_\theta^F(\chi, \psi) = \gamma_{\theta'}^F(\chi, \psi) = \gamma^F(\mu\chi, \psi)\gamma^F(\nu\chi, \psi)$, we have $\mu\nu^{-1} = \eta_{K/F}$ in both cases; so $\theta = \mu\circ N_{K/F} = \nu\circ N_{K/F}$ and $\theta\circ N_{B/K} = (\mu\otimes\nu)\circ N_{B/K'}$, which means that θ and θ' have the same lift to $B^\times$.

We have proved the "necessary" part of the theorem. If K and K' are isomorphic and θ' corresponds to θ or $\overline{\theta}$, then $\lambda_{K/F}(\psi) = \lambda_{K'/F}(\psi)$ and $\gamma_\theta^F = \gamma_{\overline{\theta}}^F$ since both $\chi\circ N$ and $\psi\circ T$ are invariant by the conjugation of K over F. Assume now B is a field, and that $\theta\circ N_{B/K} = \theta'\circ N_{B/K'}$ with $\theta|_{F^\times} \neq \theta'|_{F^\times}$. We use the third quadratic subextension K'' of B, and write $\theta\circ N_{B/K} = \theta'\circ N_{B/K'}$ on $K''^\times$: this shows that the restrictions to $N_{K''/F}K''^\times$, which is an index two subgroup of $F^\times$, of θ and θ' are equal. Hence $\theta'|_{F^\times} = \eta_{K''/F}\theta|_{F^\times}$, that is, $\omega = \omega'$. We prove that this implies that θ and θ' are regular over F. Indeed, if $\theta = \mu\circ N_{K/F}$ then $\theta\circ N_{B/K} = \mu\circ N_{K'/F}\circ N_{B/K'} = \theta'\circ N_{B/K'}$; hence $\mu\circ N_{K'/F}$ and θ' coincide on the elements of $K'^\times$ which are norms from $B^\times$, in particular on $F^\times$: this contradicts $\theta|_{F^\times} \neq \theta'|_{F^\times}$. The relation $\theta \circ N_{B/K} = \theta' \circ N_{B/K'}$ shows that this character of $B^\times$ is fixed by both $\operatorname{Gal} B/K$ and $\operatorname{Gal} B/K'$, hence by all $\operatorname{Gal} B/F$, that is, also by $\operatorname{Gal} B/K''$. This shows that θ and its conjugate $\overline{\theta}$ over F have the same lift to $B^\times$, and θ being regular, this gives $\overline{\theta} = \theta\eta_{B/K}$, and also $\overline{\theta}' = \theta'\eta_{B/K'}$. As $\eta_{B/K}$ is the lift of $\eta_{K''/F}$ to $K^\times$, the relation $\gamma_\theta^F = \gamma_{\overline{\theta}}^F$ shows that $\gamma_\theta^F(\chi, \psi) = \gamma_\theta^F(\chi\eta_{K''/F}, \psi)$, and also for $\gamma_{\theta'}^F$. We apply Theorem 1 to get the relation $\langle \gamma_{\theta''}^F | \gamma_\theta^F \rangle_\psi = \langle \gamma_{\theta''}^F | \gamma_{\theta'}^F \rangle_\psi$ for $\theta'' \in \mathscr{A}(K''^\times)$, with a finite number of exceptions. We take $\theta'' = \chi \circ N_{K''/F}$ and use the Davenport-Hasse identity to get

$$\gamma_\theta^F(\chi, \psi)\gamma_\theta^F(\chi\eta_{K''/F}, \psi) = \gamma_{\theta'}^F(\chi, \psi)\gamma_{\theta'}^F(\chi\eta_{K''/F}, \psi),$$

that is, $\gamma_\theta^F(\chi, \psi)^2 = \gamma_{\theta'}^F(\chi, \psi)^2$. We define a sign $\varepsilon(\chi)$ by

$$\gamma_{\theta'}^F(\chi, \psi) = \varepsilon(\chi)\gamma_\theta^F(\chi, \psi), \qquad \chi \in \mathscr{A}(F^\times).$$

This sign is 1 for χ with large conductor, due to the deep twist property and $\omega = \omega'$. As θ and θ' are regular, the rational functions $\gamma_\theta^F(\chi, \psi)$ and $\gamma_{\theta'}^F(\chi, \psi)$ are monomials on $\mathscr{A}(F^\times)$, and having the same squares, the degree is the same, as $\varepsilon(\chi)$ is constant on each component of $\mathscr{A}(F^\times)$. Theorem 1 shows that $\langle \gamma_{\theta'}^F | \gamma_{\theta^{-1}}^F \rangle_\psi = 0$, hence, for M large enough,

$$\begin{aligned} 0 &= \oint_{a(\chi)\le M} \gamma_{\theta'}^F(\chi, \psi)\gamma_{\theta^{-1}}^F(\chi^{-1}, \psi)\,d\chi - q^M(1 - q^{-1}) \\ &= \left(\oint_{a(\chi)\le M} \varepsilon(\chi)\,d\chi - q^M(1-q^{-1})\right)\omega(-1), \end{aligned}$$

that is, $\sum_{a(\chi)\le M} \varepsilon(\chi) = \sum_{a(\chi)\le M} 1$, the summations being on the characters of $\mathscr{O}^\times$ with $a(\chi)$ at most M. As $\varepsilon(\chi)$ is a sign, this implies $\varepsilon(\chi) = 1$ for all χ's, and $\gamma_\theta^F = \gamma_{\theta'}^F$. The theorem is completely proved.

Corollary. *With K, K', θ, θ' as in the theorem, we have $\gamma_\theta^F = \gamma_{\theta'}^F$ if and only if $\operatorname{Ind}_K^F \theta = \operatorname{Ind}_{K'}^F \theta'$.*

Proof. (a) This is clear if K and K' are isomorphic since $\operatorname{Ind}_K^F \theta$ and $\operatorname{Ind}_{K'}^F \theta'$ are the same if and only if $\theta' = \theta$ or $\overline{\theta}$, which means $\gamma_\theta^F = \gamma_{\theta'}^F$ by Theorem 4.

(b) Assume K is a field and K' is $F \times F$. Write $\theta' = \mu \otimes \nu$, so that $\operatorname{Ind}_{K'}^F \theta'$ is the direct sum of the two one-dimensional representations μ and ν of $F^\times$, abelianized group of W_F. Theorem 4 shows that $\gamma_\theta^F = \gamma_{\theta'}^F$ if and only if $\theta = \mu \circ N_{K/F}$ and $\nu = \mu\eta_{K/F}$; then $\operatorname{Ind}_K^F \theta = \mu \otimes \operatorname{Ind}_K^F 1 = \mu \oplus \mu\eta_{K/F} = \mu \oplus \nu = \operatorname{Ind}_{K'}^F \mu \otimes \nu = \operatorname{Ind}_{K'}^F \theta'$. Conversely, if $\operatorname{Ind}_K^F \theta$ is the sum of the two characters μ and ν, then $\operatorname{Ind}_K^F \theta$ is not irreducible, so $\theta = \chi \circ N_{K/F}$; but then $\operatorname{Ind}_K^F \theta = \chi \oplus \chi\eta_{K/F}$ so that $\chi = \mu$ or ν and $\nu = \mu\eta_{K/F}$, and the theorem says $\gamma_\theta^F = \gamma_{\theta'}^F$.

(c) Assume now $B = K \otimes K'$ is a field. If $\gamma_\theta^F = \gamma_{\theta'}^F$, then, by Theorem 4, we have $\theta \circ N_{B/K} = \theta' \circ N_{B/K'}$ and $\theta|_{F^\times} \neq \theta'|_{F^\times}$; during the proof, we have shown that this implies $\overline{\theta} = \theta\eta_{B/K}$ and $\overline{\theta}' = \theta'\eta_{B/K'}$. As the trace of $\mathrm{Ind}_K^F \theta$ is 0 outside W_K and on W_K it factors through its abelianization $K^\times$ where it is given by $(\theta + \overline{\theta})/2$, this is 0 outside W_B and on W_B it factors through its abelianization $B^\times$ where it is given by $\theta \circ N_{B/K}$. Hence $\mathrm{Ind}_K^F \theta$ and $\mathrm{Ind}_{K'}^F \theta'$ have their traces supported on W_B, where they are equal, so the representations $\mathrm{Ind}_K^F \theta$ and $\mathrm{Ind}_{K'}^F \theta'$ are equivalent. Conversely, this equivalence implies that traces and determinants of two representations are the same; for the determinants, this gives $\theta|_{F^\times}\eta_{K/F} = \theta'|_{F^\times}\eta_{K'/F}$, so $\theta|_{F^\times}$ and $\theta'|_{F^\times}$ are different; for the traces, we get 0 outside $W_K \cap W_{K'} = W_B$, that is, $\overline{\theta} = \theta\eta_{B/K}$, $\overline{\theta}' = \theta'\eta_{B/K'}$, and the coincidence on W_B says that $\theta \circ N_{B/K} = \theta' \circ N_{B/K'}$. Theorem 1 then concludes the proof.

References

[D] P. Deligne, *Les constantes des équations fonctionnelles des fonctions* L, Modular Functions in One Variable. II, Lecture Notes in Math., vol. 349, Springer-Verlag, Berlin-Heidelberg-New York, 1973, pp. 501–597.

[G] P. Gérardin, *Groupes quadratiques et applications arithmétiques*, Séminaire de Théorie des Nombres, Paris, 1984–85 (C. Golstein, Ed.), Progress in Math., Vol. 63, Birkhäuser, Boston, Mass., 1986, pp. 79–95.

[GK] P. Gérardin and P. C. Kutzko, *Facteurs locaux pour* GL(2), Ann. Sci. École Norm Sup. (4) **13** (1980), 359–384.

[GL1] P. Gérardin and W.-C. W. Li, *A functional equation for degree two local factors*, Canad. Math. Soc. Bull. **28** (1985), 355–371.

[GL2] ——, *Fourier transforms of representations of quaternions*, J. Reine Angew. Math. **349** (1985), 121–173.

[GL3] ——, *Degree two monomial representations of local Weil groups*, J. Reine Angew. Math. 394 (1989), 1–30.

[L] W.-C. W. Li, *Barnes' identities and representations of* GL_2. *Part* II: *Nonarchimedean local field case*, J. Reine Angew. Math. **345** (1983), 69–92.

[S] J.-P. Serre, *Corps locaux*, Hermann, Paris, 1964.

[W1] A. Weil, *Sur certains groupes d'opérateurs unttaires*, Acta Math. **111** (1964), 153–211.

[W2] ——, *Basic number theory*, Springer-Verlag, Berlin and New York, 1967.

[W3] ——, *Fonctions zeta et distributions*, Séminaire Bourbaki, Juin 1966.

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